Face-to-face Interactions of Breathers in Electron-Ion Plasma with the Critical Composition of an Electron Beam

Prasanta Chatterjee[1] **and Jayshree Mondal**[2]

Department of Mathematics, Visva Bharati University, Bolpur, Santiniketan, West Bengal- 731235, India

E-mail: [1]**prasantacvb@gmail.com**; [2]**jayshreemondal2512@gmail.com**

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Abstract: In the present work, the collisions of weakly nonlinear wave structures in an unmagnetized plasma comprising inertial ions, Maxwellian distributed electrons, and electron beams are analyzed based on experimental data. The study focuses on face-to-face collisions involving breather–breather interaction. To carry out this analysis, the extended Poincaré–Lighthill–Kuo (PLK) reductive perturbation method is employed, leading to the derivation of two coupled Korteweg–de Vries (KdV) equations. At a critical value of the number density ratio, two coupled modified Korteweg–de Vries (mKdV) equations are used to study breather–breather interaction of the ion-acoustic wave. The effects of the electron beam-to-ion density ratio (ρ_b) at equilibrium—on the colliding breather profiles are examined. The plasma parameter values used in the analysis are taken from the interplanetary space environment.

Keywords: Head-on Collision, Electron Beam, PLK Method, mKdV Equation, Breather.

I Introduction

Numerous satellite observations have confirmed the presence of nonlinear structures in space and astrophysical plasmas (SAEs). Among these, ion-acoustic (IA) waves are widely studied nonlinear excitations, investigated theoretically and experimentally across various plasma environments [1–8]. The KdV equation, first introduced by Washimi & Taniuti [1] and later verified experimentally by Ikezi et al. [2], describes IA solitons. In plasmas with negative ions, the nonlinear coefficient of KdV may change sign, allowing compressive or rarefactive solitons [9,10]. Near critical plasma parameters, nonlinearity vanishes and higher-order effects lead to the modified KdV (mKdV) equation [11–13]. Electron beams are frequently observed in plasmas containing multiple electron populations, and their presence significantly influences IA soliton properties [14–26]. Various studies have shown that beam temperature, beam density, and ion temperature play crucial roles in determining the existence and characteristics of IA solitons.

Soliton collisions are important nonlinear processes observed in optics, Bose–Einstein condensates, and plasmas [27–29]. Zabusky and Kruskal [30] first predicted soliton interactions within the KdV framework. Experiments have demonstrated that solitons preserve their shapes after collisions, undergoing only phase shifts. Two types of collisions are possible: overtaking (same direction) and head-on (opposite direction). Overtaking collisions have been investigated using inverse scattering and Hirota methods [31–34]. Head-on collisions have been extensively studied using the extended PLK method in multi-component plasmas [35–39]. For plasmas supporting both positive and negative solitons, coupled mKdV equations are required near critical conditions [40,41]. In this work, we assume a Maxwellian velocity distribution for electrons, which is suitable for laboratory and many space plasma conditions where thermal equilibrium holds. Furthermore, recent studies have reported breather-type structures in plasmas [42–46]. Very recently, breather–soliton and breather–breather interactions have been explored in plasma systems [47,48]. To the best of our knowledge, head-on collisions of breather with breather have not yet been reported in this plasma model.

This paper focuses on head-on collisions between two breathers in an electron-beam plasma. In Section II, the plasma model is presented. Section III derives the two-sided KdV and mKdV equations using the extended PLK method and obtains breather solutions through Hirota's method. Section IV analyzes parameter effects and breather–breather interactions. Section V provides conclusions.

II Mathematical Model

The nonlinear propagation of ion-acoustic solitary waves and breather waves has been studied in an impact-free, unmagnetized plasma consisting of fluid ions and electrons obeying the Maxwelian distribution, penetrated by an electron beam. The basic set of normalised equations for the ions and electron beam, that govern the dynamics of IASWs are written as follows:

$$\begin{aligned}
&\frac{\partial n_i}{\partial t}+\frac{\partial n_i u_i}{\partial x}+\frac{\partial n_i v_i}{\partial y}=0,\\
&\frac{\partial n_b}{\partial t}+\frac{\partial n_b u_b}{\partial x}+\frac{\partial n_b v_b}{\partial y}=0,\\
&\frac{\partial u_i}{\partial t}+u_i\frac{\partial u_i}{\partial x}+v_i\frac{\partial u_i}{\partial y}=-\frac{\partial \phi}{\partial x},\\
&\frac{\partial u_b}{\partial t}+u_b\frac{\partial u_b}{\partial x}+v_b\frac{\partial u_b}{\partial y}=\frac{1}{\mu_e}\frac{\partial \phi}{\partial x},\\
&\frac{\partial v_i}{\partial t}+u_i\frac{\partial v_i}{\partial x}+v_i\frac{\partial v_i}{\partial y}=-\frac{\partial \phi}{\partial y},\\
&\frac{\partial v_b}{\partial t}+u_b\frac{\partial v_b}{\partial x}+v_b\frac{\partial v_b}{\partial y}=\frac{1}{\mu_e}\frac{\partial \phi}{\partial y},
\end{aligned} \tag{1}$$

$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=-n_i+\alpha n_e+\rho_b n_b, \tag{2}$$

where $n_{j(j=i,e,b)}$ represents the number density of ions, superthermal electrons and the electron beam, respectively. The electron-to-ion mass ratio is denoted by μ_e (i.e., $\mu_e=\frac{m_e}{m_i}$). The mean velocities of the ions and the beam are represented by u_i and u_b, respectively. The electrostatic potential is denoted by ϕ. Here, charge neutrality condition $1=\alpha+\rho_b$, $\alpha=\frac{n_{e0}}{n_{i0}}$, $\rho_b=\frac{n_{b0}}{n_{i0}}$. The electron density expressed by Maxwellian distribution is written as $n_e=e^{\phi}$. The ion and beam number densities, denoted as n_i and n_b, are normalized by their respective unperturbed values, n_{i0} and n_{b0}. The electrostatic potential ϕ is normalized by T_e/e, where T_e is the electron temperature and e is the elementary charge. The fluid velocities u_i and u_b are normalized by the ion-acoustic sound speed $C_s(=(K_BT_e/m_i)^{1/2})$, where m_i is the ion mass. The space and time variables are normalized by the ion Debye length $\lambda_{D,i}=\left(\frac{K_BT_e}{4\pi n_{i0}e^2}\right)^{1/2}$ and and the ion plasma period $\omega_{pi}^{-1}=\left(\frac{m_i}{4\pi n_{i0}e^2}\right)^{1/2}$, respectively.

III Derivation of Both Sided mKdV Equation

To study the soliton interaction, we consider two weakly non-linear solitary waves S_1 and S_2 propagating in a plasma medium. These waves are initially well separated in space and approach each other from opposite directions with equal phase velocities. After some time, the waves interact, undergo a collision, and then separate. Since the interaction between them is weak, the collision is expected to be quasi-elastic—causing only a shift in their trajectories (known as a phase shift) without altering their overall shapes. To systematically analyze the effects of this interaction, we employed the extended Poincaré-Lighthill-Kuo (PLK) perturbation technique. Within this framework, the dependent variables are expanded as asymptotic series in terms of a small parameter ϵ, characterizing the wave amplitude, in the form [40]

$$\Psi = \Psi_0 + \sum_{n=1}^{\infty} \epsilon^n \Psi^n, \tag{3}$$

where $\Psi = (n_i, n_b, u_i, u_b, v_i, v_b, \phi)$ with $\Psi_0 = (1, \rho_b, 0, 0, 0, 0, 0)$. Now introducing the stretched independent variable as

$$\begin{aligned} \xi &= \epsilon(lx + my - \lambda t) + \epsilon^2 P(\eta, \tau) + \cdots, \\ \eta &= \epsilon(lx + my + \lambda t) + \epsilon^2 Q(\xi, \tau) + \cdots, \\ \tau &= \epsilon^3 t, \end{aligned} \tag{4}$$

where ξ and η denote the trajectories of the two solitary waves which are travelling toward each other. The variables P, Q, etc. will be determined later. Introducing the operators

$$\begin{aligned} \hat{X} &= \partial_\xi + \partial_\eta \\ \hat{X}' &= (\partial_\eta P)\partial_\xi + (\partial_\xi Q)\partial_\eta \\ \hat{T} &= -\partial_\xi + \partial_\eta \\ \hat{T}' &= (\partial_\eta P)\partial_\xi - (\partial_\xi Q)\partial_\eta \end{aligned} \tag{5}$$

Hence

$$\begin{aligned} \partial_t &= \lambda\epsilon\hat{T} + \lambda\epsilon^3\hat{T}' + \epsilon^3\partial_\tau + \cdots \\ \partial_x &= l\epsilon\hat{X} + l\epsilon^3\hat{X}' + \cdots \\ \partial_y &= m\epsilon\hat{X} + m\epsilon^3\hat{X}' + \cdots \end{aligned} \tag{6}$$

Substituting equations. (3) and (4) into model equations (1) and using equation (6) we obtain the ϵ independent term from equation (2) as $1 = \alpha + \rho_b$, which is the charge neutrality condition.
The coefficients of equal powers at the lowest order yield

$$\begin{aligned} \lambda\hat{T}n_i^{(1)} + l\hat{X}u_i^{(1)} + m\hat{X}v_i^{(1)} &= 0 \\ \lambda\hat{T}n_b^{(1)} + l\rho_b\hat{X}u_b^{(1)} + m\rho_b\hat{X}v_b^{(1)} &= 0 \\ \lambda\hat{T}u_i^{(1)} + l\hat{X}\phi^{(1)} &= 0 \\ \lambda\hat{T}u_b^{(1)} - \frac{l}{\mu_e}\hat{X}\phi^{(1)} &= 0 \\ \lambda\hat{T}v_i^{(1)} + m\hat{X}\phi^{(1)} &= 0 \\ \lambda\hat{T}v_i^{(1)} - \frac{m}{\mu_e}\hat{X}\phi^{(1)} &= 0 \\ n_i^{(1)} - \rho_b n_b^{(1)} - \alpha\phi^{(1)} &= 0 \end{aligned} \tag{7}$$

Solving the set of equation (7) and considering the phase velocity as

$$\lambda = \sqrt{\frac{1+\rho_b^2/\mu_e}{\alpha}}, \text{ and } -4\partial^2_{\xi\eta}\phi^1 = 0.$$

Hence, we obtain the relations among the physical quantities as

$$\begin{aligned} &\phi^{(1)} = \phi_\xi^{(1)} + \phi_\eta^{(1)}, \quad \phi_\xi^{(1)} = \phi_\xi^{(1)}(\xi, \tau), \quad \phi_\eta^{(1)} = \phi_\eta^{(1)}(\eta, \tau), \\ &n_i^{(1)} = \frac{1}{\lambda^2}(\phi_\xi^{(1)} + \phi_\eta^{(1)}), \qquad n_b^{(1)} = -\frac{\rho_b}{\mu_e\lambda^2}(\phi_\xi^{(1)} + \phi_\eta^{(1)}), \\ &u_i^{(1)} = \frac{l}{\lambda}(\phi_\xi^{(1)} - \phi_\eta^{(1)}), \qquad u_b^{(1)} = -\frac{l}{\mu_e\lambda}(\phi_\xi^{(1)} - \phi_\eta^{(1)}), \\ &v_i^{(1)} = \frac{m}{\lambda}(\phi_\xi^{(1)} - \phi_\eta^{(1)}), \qquad v_b^{(1)} = -\frac{m}{\mu_e\lambda}(\phi_\xi^{(1)} - \phi_\eta^{(1)}). \end{aligned} \tag{8}$$

At the next higher order of ϵ we have

$$\begin{aligned}\left[\frac{\alpha\lambda^2}{2}-\frac{3}{2\lambda^2}+\frac{3\rho_b^2}{2\lambda^2\mu_e^2}\right](\phi_\xi^{(1)})^2=0\\ \left[\frac{\alpha\lambda^2}{2}-\frac{3}{2\lambda^2}+\frac{3\rho_b^2}{2\lambda^2\mu_e^2}\right](\phi_\eta^{(1)})^2=0\end{aligned} \tag{9}$$

Now from couple of equation (9) two cases are arises such as

Case-1 $\phi_\xi^{(1)}=\phi_\eta^{(1)}=0.$

Case-2 $\frac{\alpha\lambda^2}{2}-\frac{3}{2\lambda^2}+\frac{3\rho_b^2}{2\lambda^2\mu_e^2}=0.$

And take $\tilde{\phi}^{(2)}=\phi^{(2)}-\phi_\xi^{(2)}-\phi_\eta^{(2)}$, $\tilde{\phi}^{(2)}$ depending on both ξ and η. Hence

$$\begin{aligned}&\frac{\partial^2}{\partial\xi\partial\eta}\left[-4\alpha\lambda^2\tilde{\phi}^{(2)}-\left(2\alpha\lambda^2-\frac{2}{\lambda^2}+\frac{2\rho_b^2}{2\lambda^2\mu_e^2}\right)\phi_\xi^{(1)}\phi_\eta^{(1)}\right]\\ &+\left[\left(\alpha\lambda^2+\frac{1}{\lambda^2}-\frac{\rho_b^2}{\lambda^2\mu_e^2}\right)\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\phi_\xi^{(1)}\phi_\eta^{(1)}\right]=0\end{aligned} \tag{10}$$

III.a For Case-1:

$\phi_\xi^{(1)}=\phi_\eta^{(1)}=0$, then we have the following relation

$$\begin{aligned}&\phi^{(2)}=\phi_\xi^{(2)}+\phi_\eta^{(2)}, \quad \tilde{\phi}^{(2)}=0,\\ &n_i^{(2)}=\frac{1}{\lambda^2}(\phi_\xi^{(2)}+\phi_\eta^{(2)}), \qquad n_b^{(2)}=-\frac{\rho_b}{\mu_e\lambda^2}(\phi_\xi^{(2)}+\phi_\eta^{(2)}),\\ &u_i^{(2)}=\frac{l}{\lambda}(\phi_\xi^{(2)}-\phi_\eta^{(2)}), \qquad u_b^{(2)}=-\frac{l}{\mu_e\lambda}(\phi_\xi^{(2)}-\phi_\eta^{(2)}),\\ &v_i^{(2)}=\frac{m}{\lambda}(\phi_\xi^{(2)}-\phi_\eta^{(2)}), \qquad v_b^{(2)}=-\frac{m}{\mu_e\lambda}(\phi_\xi^{(2)}-\phi_\eta^{(2)}).\end{aligned} \tag{11}$$

We have the same result for the next higher order of ϵ.

Furthermore, by considering the next higher order of ϵ, we lead to

$$\begin{aligned}&\frac{2}{\alpha}\left(lu_i^{(4)}+mv_i^{(4)}-l\rho_b^2u_b^{(4)}-m\rho_b^2v_b^{(4)}\right)\\ &=-\int\left(\frac{\partial\phi_\xi^{(2)}}{\partial\tau}-A\phi_\xi^{(2)}\frac{\partial\phi_\xi^{(2)}}{\partial\xi}+B\frac{\partial^3\phi_\xi^{(2)}}{\partial\xi^3}\right)\mathrm{d}\eta\\ &\quad-\int\left(\frac{\partial\phi_\eta^{(2)}}{\partial\tau}+A\phi_\eta^{(2)}\frac{\partial\phi_\eta^{(2)}}{\partial\eta}-B\frac{\partial^3\phi_\eta^{(2)}}{\partial\eta^3}\right)\mathrm{d}\xi\\ &\quad+\int\int\left(C\frac{\partial P_0}{\partial\eta}-D\phi_\eta^{(2)}\right)\frac{\partial\phi_\xi^{(2)}}{\partial\xi^2}\mathrm{d}\xi\mathrm{d}\eta\\ &\quad-\int\int\left(C\frac{\partial Q_0}{\partial\xi}-D\phi_\xi^{(2)}\right)\frac{\partial\phi_\eta^{(2)}}{\partial\eta^2}\mathrm{d}\xi\mathrm{d}\eta,\end{aligned} \tag{12}$$

where

$$\begin{aligned}&A=\frac{1}{\alpha\lambda}\left[\frac{\alpha\lambda^2}{2}-\frac{3}{2\lambda^2}+\frac{3\rho_b^2}{2\lambda^2\mu_e^2}\right], \quad B=\frac{\lambda}{2\alpha},\\ &C=2\lambda, \quad D=\frac{1}{2\alpha\lambda}\left[\alpha\lambda^2+\frac{1}{\lambda^2}-\frac{\rho_b^2}{\lambda^2\mu_e^2}\right].\end{aligned} \tag{13}$$

The first(second) term in the right hand side of equation (12) will be proportional to $\eta(\xi)$ as the integrated function is independent of $\eta(\xi)$. In order to avoid spurious resonance, the first two secular terms of equation (12) must be eliminated. So we have the typical KdV equations for the right and left going solitary waves, respectively, as

$$\begin{aligned} \frac{\partial \phi_\xi^{(2)}}{\partial \tau} - A\phi_\xi^{(2)}\frac{\partial \phi_\xi^{(2)}}{\partial \xi} + B\frac{\partial^3 \phi_\xi^{(2)}}{\partial \xi^3} = 0, \\ \frac{\partial \phi_\eta^{(2)}}{\partial \tau} + A\phi_\eta^{(2)}\frac{\partial \phi_\eta^{(2)}}{\partial \eta} - B\frac{\partial^3 \phi_\eta^{(2)}}{\partial \eta^3} = 0, \end{aligned} \tag{14}$$

with $A = \frac{1}{\alpha\lambda}\left[\frac{\alpha\lambda^2}{2} - \frac{3}{2\lambda^2} + \frac{3\rho_b^2}{2\lambda^2\mu_e^2}\right], \quad B = \frac{\lambda}{2\alpha}$. The last two terms in equation (12) are not secular in this order, but they could be secular in the next order. Hence we have

$$C\frac{\partial P_0}{\partial \eta} - D\phi_\eta^{(2)} = 0, \quad C\frac{\partial Q_0}{\partial \xi} - D\phi_\xi^{(2)} = 0. \tag{15}$$

Equations (14) are two-sided travelling wave KdV equations in the reference frames of ξ and η respectively. The well-known "sech-squared" solutions are as follows

$$\begin{aligned} \phi_\xi^{(2)} = -\frac{12B}{A}\text{Sech}^2\left(\xi - 4B\tau\right), \\ \phi_\eta^{(2)} = -\frac{12B}{A}\text{Sech}^2\left(\xi + 4B\tau\right). \end{aligned} \tag{16}$$

III.b For Case-2:

If $\frac{\alpha\lambda^2}{2} - \frac{3}{2\lambda^2} + \frac{3\rho_b^2}{2\lambda^2\mu_e^2} = 0$ then $A = 0$. For $A = 0$, the quadratic non-linearity in the KdV equation (14) will disappear, and the cubic nonlinearity will appear. Then from equation (10) it is clear that $\tilde{\phi}^{(2)} \neq 0$, we can cancel [35] the solutions of the linear operator without loss of generality, hence, $\phi_\xi^{(2)} = 0 = \phi_\eta^{(2)}$ and then the solution is of the form

$$\begin{aligned} &\phi^{(2)} = \tilde{\phi}_\xi^{(2)}, \quad \tilde{n}_{ibp}^{(2)} = n_{ibp}^{(2)} - n_{i,b}^{(2)}(\xi,\tau) - n_{i,b}^{(2)}(\xi,\tau) \\ &\tilde{u}_{i,b}^{(2)} = u_{i,b}^{(2)} - u_{i,b}^{(2)}(\xi,\tau) - u_{i,b}^{(2)}(\xi,\tau), \\ &\tilde{v}_{i,b}^{(2)} = v_{i,b}^{(2)} - v_{i,b}^{(2)}(\xi,\tau) - v_{i,b}^{(2)}(\xi,\tau) \\ &n_i^{(2)}(\xi,\tau) = \frac{3}{2\lambda^4}(\phi_\xi^{(1)})^2, \qquad n_i^{(2)}(\eta,\tau) = \frac{3}{2\lambda^4}(\phi_\eta^{(1)})^2, \\ &n_b^{(2)}(\xi,\tau) = \frac{3\rho_b}{2\lambda^4\mu_e^2}(\phi_\xi^{(1)})^2, \quad n_b^{(2)}(\eta,\tau) = \frac{3\rho_b}{2\lambda^4\mu_e^2}(\phi_\eta^{(1)})^2, \\ &u_i^{(2)}(\xi,\tau) = \frac{l}{2\lambda^3}(\phi_\xi^{(1)})^2, \qquad u_i^{(2)}(\eta,\tau) = -\frac{l}{2\lambda^3}(\phi_\eta^{(1)})^2, \\ &u_b^{(2)}(\xi,\tau) = \frac{l}{2\lambda^3\mu_e^2}(\phi_\xi^{(1)})^2, \quad u_b^{(2)}(\eta,\tau) = -\frac{l}{2\lambda^3\mu_e^2}(\phi_\eta^{(1)})^2, \\ &v_i^{(2)}(\xi,\tau) = \frac{m}{2\lambda^3}(\phi_\xi^{(1)})^2, \qquad v_i^{(2)}(\eta,\tau) = -\frac{m}{2\lambda^3}(\phi_\eta^{(1)})^2, \\ &v_b^{(2)}(\xi,\tau) = \frac{m}{2\lambda^3\mu_e^2}(\phi_\xi^{(1)})^2, \quad v_b^{(2)}(\eta,\tau) = -\frac{m}{2\lambda^3\mu_e^2}(\phi_\eta^{(1)})^2. \end{aligned} \tag{17}$$

Now, on the next order of ϵ, we have

$$
\begin{aligned}
&-\frac{2}{\alpha}\left(lu_i^{(3)}+mv_i^{(3)}-l\rho_b^2u_b^{(3)}-m\rho_b^2v_b^{(3)}\right)\\
&=\int\left(\frac{\partial\phi_\xi^{(1)}}{\partial\tau}+A(\phi_\xi^{(1)})^2\frac{\partial\phi_\xi^{(1)}}{\partial\xi}+B\frac{\partial^3\phi_\xi^{(1)}}{\partial\xi^3}\right)\mathrm{d}\eta\\
&+\int\left(\frac{\partial\phi_\eta^{(1)}}{\partial\tau}-A(\phi_\eta^{(1)})^2\frac{\partial\phi_\eta^{(1)}}{\partial\eta}-B\frac{\partial^3\phi_\eta^{(1)}}{\partial\eta^3}\right)\mathrm{d}\xi\\
&+\int\int\left(C\frac{\partial P_0}{\partial\eta}-D(\phi_\eta^{(1)})^2\right)\frac{\partial\phi_\xi^{(1)}}{\partial\xi^2}\mathrm{d}\xi\mathrm{d}\eta\\
&-\int\int\left(C\frac{\partial Q_0}{\partial\xi}-D(\phi_\xi^{(1)})^2\right)\frac{\partial\phi_\eta^{(1)}}{\partial\eta^2}\mathrm{d}\xi\mathrm{d}\eta+R',
\end{aligned}
\tag{18}
$$

where

$$
\begin{aligned}
A&=\left[\frac{15}{2\lambda^4}+\frac{15\rho_b^2}{2\lambda^4\mu_e^3}-\frac{\alpha\lambda^2}{2}\right]\times\frac{1}{2\alpha\lambda},\\
B&=\frac{\lambda}{2\alpha},\qquad C=2\lambda,\ \text{and}\\
D&=\left[\frac{\alpha\lambda^2}{2}-\frac{1}{2\lambda^4}-\frac{\rho_b^2}{2\lambda^4\mu_e^3}\right]\times\frac{1}{2\alpha\lambda}.
\end{aligned}
\tag{19}
$$

By combining the parts of these equations that contain terms dependent on ξ or η (in addition to τ), we obtain the standard modified Korteweg–de Vries (mKdV) equations for the right and left propagating solitary waves, respectively, as

$$
\frac{\partial\phi_\xi^{(1)}}{\partial\tau}+A(\phi_\xi^{(1)})^2\frac{\partial\phi_\xi^{(1)}}{\partial\xi}+B\frac{\partial^3\phi_\xi^{(1)}}{\partial\xi^3}=0,
\tag{20}
$$

$$
\frac{\partial\phi_\eta^{(1)}}{\partial\tau}-A(\phi_\eta^{(1)})^2\frac{\partial\phi_\eta^{(1)}}{\partial\eta}-B\frac{\partial^3\phi_\eta^{(1)}}{\partial\eta^3}=0.
\tag{21}
$$

The 3rd and 4th terms in equation (18) are not secular at this order, but they may become secular in the next order. Hence we have

$$
C\frac{\partial P_0}{\partial\eta}-D(\phi_\eta^{(1)})^2=0,\quad C\frac{\partial Q_0}{\partial\xi}-D(\phi_\xi^{(1)})^2=0
\tag{22}
$$

III.c Solution of mKdV Equation:

Here, we use the Hirota bilinear method to solve the mKdV equation. The transformation

$$
\phi_\xi^{(1)}=\sqrt{\frac{24B}{A}}\frac{\partial}{\partial\xi}\left(\tan^{-1}\frac{g_1}{f_1}\right)
$$

is employed to obtain the Hirota bilinear form of Equation (20). The resulting Hirota bilinear form of Equation (20) is given by:

$$
\left(D_\tau+BD_\xi^3\right)(g_1.f_1)=0,
\tag{23}
$$

$$
D_\xi^2\left(f_1.f_1+g_1.g_1\right)=0.
\tag{24}
$$

Similarly, by applying the transformation

$$
\phi_\eta^{(1)}=\sqrt{\frac{24B}{A}}\frac{\partial}{\partial\eta}\left(\tan^{-1}\frac{g_2}{f_2}\right),
$$

we obtain the Hirota bilinear form of Equation (21), which is given below

$$\left(D_\tau - BD_\eta^3\right)(g_2.f_2) = 0, \tag{25}$$

$$D_\eta^2\left(f_2.f_2 + g_2.g_2\right) = 0. \tag{26}$$

III.c.1 1-Soliton Solution:

For a 1-soliton solution, take the auxiliary functions as [49]

$$f_1 = f_0 + \epsilon F_1 \exp\theta, \quad g_1 = g_0 + G_1 \exp(\theta), \tag{27}$$

$$f_2 = f_0 + F_2 \exp(\Theta), \quad g_2 = g_0 + G_2 \exp(\Theta), \tag{28}$$

where $\theta = k_1\xi - w_1\tau$ and $\Theta = l_1\eta - \omega_1\tau$. Using the values of f_1, g_1 into the equation (23) and (24) we have $w = Bk^3$, $\omega = -Bl^3$, and either $F_1 = 0$ and $g_0 = 0$ or $f_0 = 0$ and $G_1 = 0$.
If we choose $F_1 = 0$ and $g_0 = 0$, and also take $f_0 = 1$ and $G_1 = 1$, then $f_1 = 1$ and $g_1 = \exp(\theta)$. Therefore, the one-soliton solution of Equation (21) is given by

$$\phi_\xi^{(1)} = k_1\sqrt{\frac{6B}{A}}\,\mathrm{sech}(\theta), \tag{29}$$

where $\theta = k_1\xi - Bk_1^3\tau$.
Similarly the one-soliton solution of equation (20) is given by

$$\phi_\eta^{(1)} = l_1\sqrt{\frac{6B}{A}}\mathrm{sech}(\Theta), \tag{30}$$

where $\Theta = l_1\eta + Bl_1^3\tau$.

III.c.2 2-Soliton Solution:

For the two-soliton solution, we take the auxiliary functions as given in [49]:

$$\begin{aligned} f_1 &= 1 + \epsilon^2 a_{12}\exp(\theta + \psi), \\ g_1 &= \epsilon\exp(\theta) + \exp(\psi), \\ f_2 &= 1 + \epsilon^2 a_{22}\exp(\Theta + \Psi), \\ g_2 &= \epsilon\exp(\Theta) + \exp(\Psi), \end{aligned} \tag{31}$$

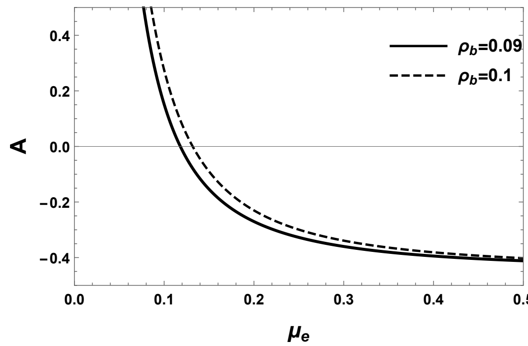
where $\theta = k_1\xi - w_1\tau$, $\psi = k_2\xi - w_2\tau$, $\Theta = l_1\eta - \omega_1\tau$ and $\Psi = l_2\eta - \omega_2\tau$. Using the values of f_1,f_2, g_1, g_2 into the equation (23)-(26) we obtain

$$w_i = Bk_i^3, \quad \omega_i = -Bl_i^3, \quad a_{12} = -\frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \quad a_{22} = -\frac{(l_1 - l_2)^2}{(l_1 + l_2)^2}. \tag{32}$$

Hence the 2-soliton solution of equation (20) and (21) are given by

$$\begin{aligned} \phi_\xi^{(1)} &= \sqrt{\frac{24B}{A}}\frac{\left(k_1e^\theta + k_2e^\psi - a_{12}e^{\theta+\psi}(k_2e^\theta + k_1e^\psi)\right)}{(1 + e^{2\theta} + e^{2\psi} + 2\left(1 + a_{12}\right)e^{\theta+\psi} + a_{12}^2e^{2\theta+2\psi})} \\ \phi_\eta^{(1)} &= \sqrt{\frac{24B}{A}}\frac{\left(l_1e^\Theta + l_2e^\Psi - a_{22}e^{\Theta+\Psi}\left(l_2e^\Theta + l_1e^\Psi\right)\right)}{(1 + e^{2\Theta} + e^{2\Psi} + 2\left(1 + a_{22}\right)e^{\Theta+\Psi} + a_{22}^2e^{2\Theta+2\Psi})}, \end{aligned} \tag{33}$$

where $\theta = k_1\xi - Bk_1\tau$, $\psi = k_2\xi - Bk_2\tau$, $\Theta = l_1\eta + Bl_1\tau$ and $\Psi = l_2\eta + Bl_2\tau$.



(a)

Figure 1: The polarity of the ion-acoustic solitons corresponding to the sign of the nonlinear coefficient A of the KdV equation.

III.d Breather Solution:

To find a breather solution, we take the complex conjugate wave numbers ($k_1 = m_1 + in_1$, $k_2 = m_1 - in_1$, $l_1 = m_2 + in_2$ and $l_2 = m_2 - in_2$) in the 2-soliton solution.
Hence, the solutions are given by

$$\phi_\xi^{(1)} = \sqrt{\frac{24B}{A}}\frac{\left(2m_1e^p\cos q - 2n_1e^p\sin q - \frac{n_1^2}{m_1^2}e^{2p}\left(2m_1e^p\cos q + 2n_1e^p\sin q\right)\right)}{\left(1 + 2e^{2p}\cos 2q + 2\left(1+\frac{n_1^2}{m_1^2}\right)e^{2p} + \frac{n_1^4}{m_1^4}e^{4p}\right)}, \tag{34}$$

$$\phi_\eta^{(1)} = \sqrt{\frac{24B}{A}}\frac{\left(2m_2e^{p_1}\cos q_1 - 2n_2e^{p_1}\sin q_1 - \frac{n_2^2}{m_2^2}e^{2p_1}\left(2m_2e^{p_1}\cos q_1 + 2n_2e^{p_1}\sin q_1\right)\right)}{\left(1 + 2e^{2p_1}\cos 2q_1 + 2\left(1+\frac{n_2^2}{m_2^2}\right)e^{2p_1} + \frac{n_2^4}{m_2^4}e^{4p_1}\right)}, \tag{35}$$

where $p = m_1\left(\xi - B(m_1^2 - 3n_1^2)\tau\right)$, $q = n_1\left(\xi - B(3m_1^2 - n_1^2)\tau\right)$, $p_1 = m_2\left(\eta + B(m_2^2 - 3n_2^2)\tau\right)$ and $q_1 = n_2\left(\eta + B(3m_2^2 - n_2^2)\tau\right)$.

IV Result and Discussion

This paper investigates the face-to-face interactions of breather with breather in a collisionless unmagnetized plasma comprising fluid ions, Maxwellian distributed electrons, and an electron beam. Using the extended PLK method, we obtained both-sided KdV and mKdV-type equations. To derive one-soliton, two-soliton, and breather structures, we use the Hirota Bilinear method [49]. We have also numerically analysed the effect of density ratio of the beam to ions ($\rho_b = n_{b0}/n_{i0}$) on the characteristics of IA solitons. The typical plasma parameters considered for the analysis, correspond to interplanetary space: $n_i \sim n_e = 5 - 10$ cm^{-3} and $n_b \sim 2$ cm^{-3}. Numerical analyses reveal the existence of two distinct types of ion-acoustic solitons (IASs): *compressive* and *rarefactive*. The nature of these solitons is determined by the sign of the coefficient of the quadratic nonlinearity A in Eqs. (14). Specifically, *positive polarity* solitons arise when $A > 0$, while *negative polarity* solitons occur for $A < 0$. This behavior is consistent with the standard KdV framework, wherein the coefficient A vanishes at a critical value of the electron-to-ion mass ratio μ_e. At this critical point, quadratic nonlinearity does not balance dispersive effects, necessitating the inclusion of *higher-order nonlinearities* to adequately describe the dynamics of the system. In Fig. 1, the coefficient A is plotted against μ_e .

In Fig. 2(a)–Fig. 2(e), we have shown the head-on collision of the breather solution of the mKdV equations. To do so we have take the parameters as $\rho_b = 0.09$, $\alpha = 0.91$, $\mu_e = 0.0005$, $m_1 = 0.5$,

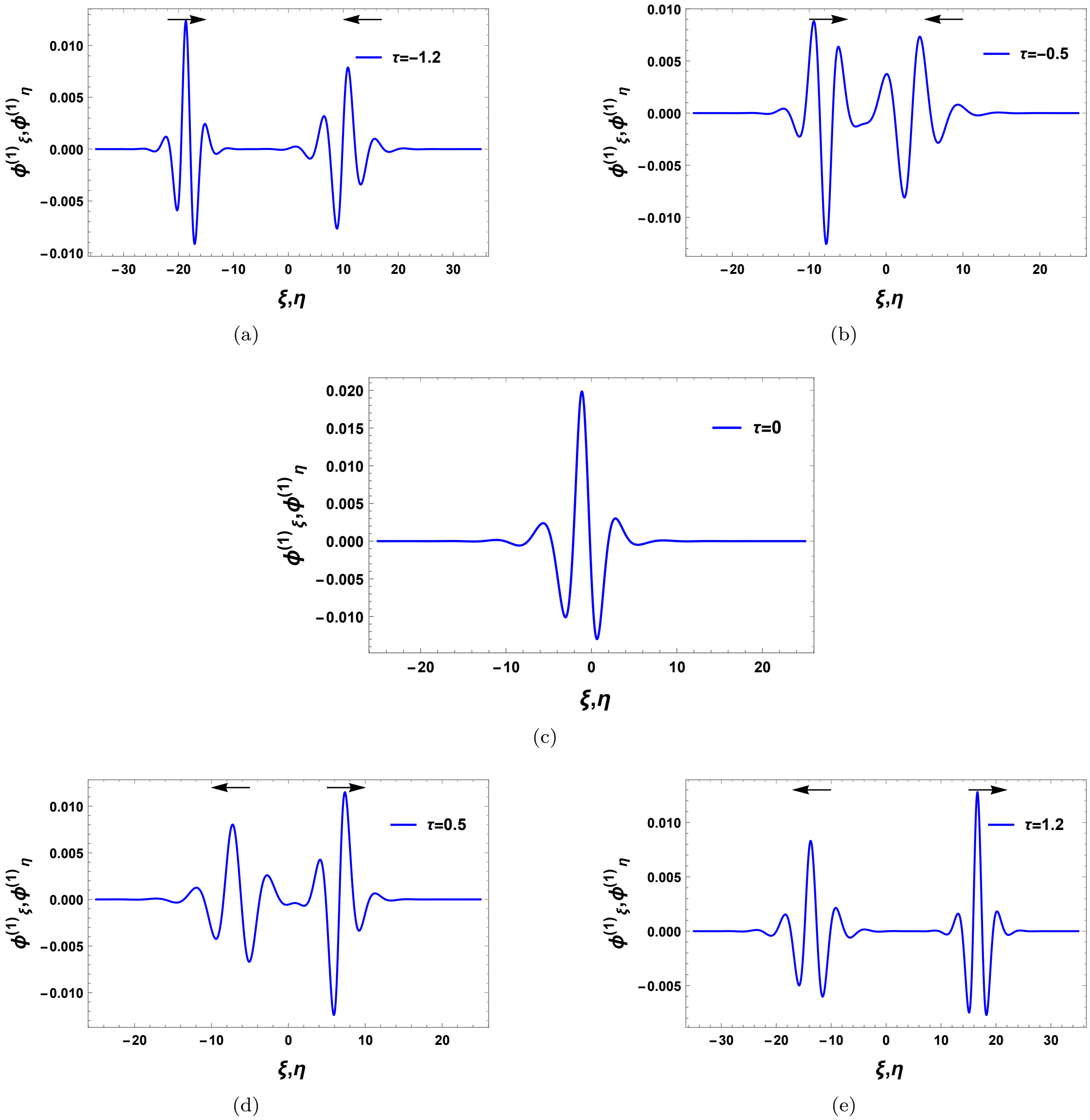
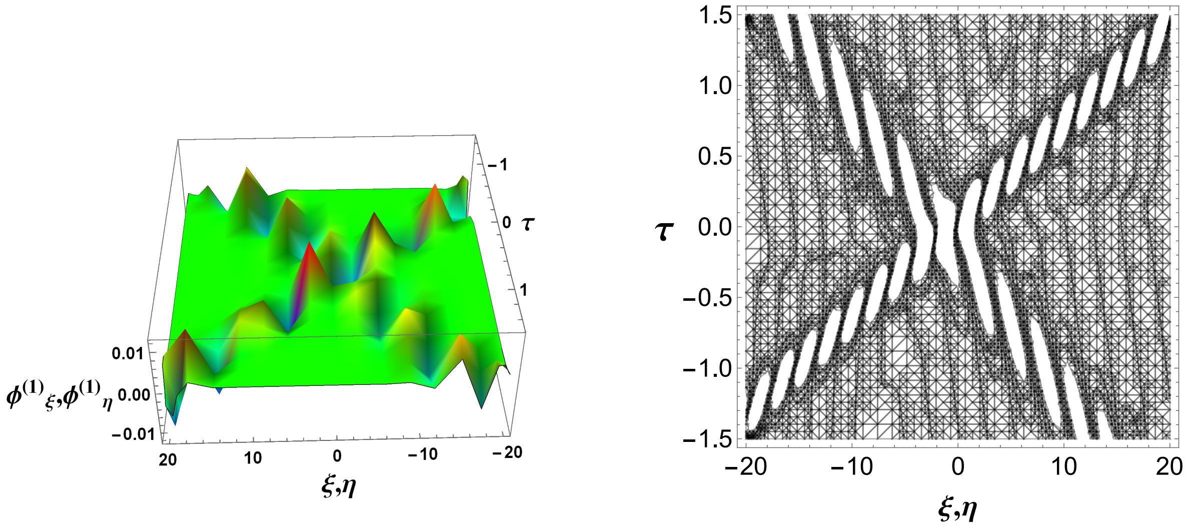


Figure 2: Collision of Breather with breather $\phi_\xi^{(1)}$ and $\phi_\eta^{(1)}$ for different values of τ with $\kappa_e = 3.5$, $\rho_b = 0.09$, $\alpha = 0.91$, $\mu_e = 0.0005$, $m_1 = 0.5$, $n_1 = 1.2$, $m_2 = 0.8$, and $n_2 = 1.5$.

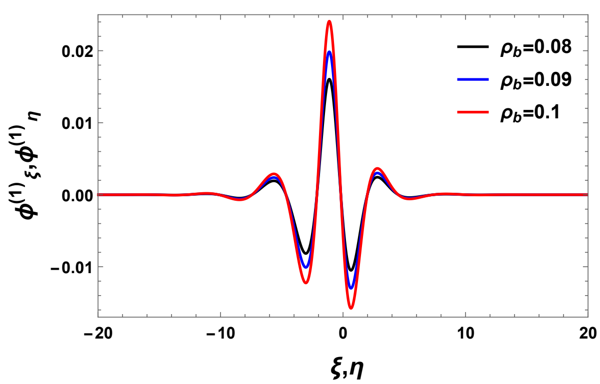
$n_1 = 1.2$, $m_2 = 0.8$, and $n_2 = 1.5$. Figure 2(a) presents the initial state at $\tau = -1.2$, where a breather propagates from left to right and another from right to left. As τ increases towards zero, the two breather structures gradually approach each other, and at $\tau = 0$, they merge into a single breather, as shown in Fig. 2(c). After the merger, they separate and regain their original shapes at $\tau = 1.2$, as illustrated in Fig. 2(e). Although it is well known that solitons recover their shapes after a head-on collision, this result demonstrates that nonlinear breather structures can also regain their shape following face-to-face interactions. This phenomenon has not been given earlier in the Plasma literature.

Additionally, Figure 3(a) provides a clear three-dimensional representation of the overlapping compressive



(a) (b)

Figure 3: 3D profile and contour profile of breather's interaction profiles $\phi_\xi^{(1)}$ and $\phi_\eta^{(1)}$ for different values of τ with $\kappa_e = 3.5$, $\rho_b = 0.09$, $\alpha = 0.91$, $\mu_e = 0.0005$, $m_1 = 0.5$, $n_1 = 1.2$, $m_2 = 0.8$, and $n_2 = 1.5$. Here 3D profile 3a and contour profile 3b



(a)

Figure 4: Effect of effect of density ratio of the beam to ions on the breathers solution of the mKdV equation during head-on collision

breather and Figure 3(b) illustrate the contour structure of HOC of breathers.

Figure 4(a)illustrate the effects of the number density ratio of electron beam to ions—on the breathers interaction of ion-acoustic waves (IAWs) within the framework of the mKdV equation. Figure 4(a) depicts the effect of the density ratio of electron beam and ions on the breathers solution of mKdV equation during head-on collision, using fixed values of $\tau = 0$, $\alpha = 0.91$, $\mu_e = 0.0005$, $m_1 = 0.5$, $n_1 = 1.2$, $m_2 = 0.8$, and $n_2 = 1.5$. The results indicate that the system energy increases (as the amplitude increases) with higher values of the electron beam number density in the plasma system. It is emphasized that the plasma parameter, density of electron beams and ions have a great influence on the breather profile of the mKdV equation during collision.

V Conclusion

This study investigated the head-on collision of breathers in an unmagnetized plasma containing fluid ions, Maxwellian electrons, and an electron beam. Using the extended PLK method, we derived mKdV equations to account for higher-order nonlinearities that arise when standard KdV coefficients vanish at critical plasma parameters.The analysis demonstrates that the electron beam-to-ion density ratio (ρ_b) significantly modifies the profiles of colliding breathers. These results provide the first detailed insight into breather dynamics within this specific plasma configuration, offering a clearer understanding of how beam-plasma interactions influence nonlinear wave stability and phase shifts.

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